MATH 3060 Tutorial 4

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1 Some problems with Homework 2 and 3

- Showing $f'(0)$ does not exist is not equivalent to showing $\lim_{x\to 0} f'(x)$ does not exist.
- $\mathbb{Q}^\infty = \mathbb{Q}\times\mathbb{Q}\times\mathbb{Q}\times\cdots$ is not countable. The set $\mathbb{Q}[x]$ is in bijection with $\cup_{k=1}^\infty \mathbb{Q}^k$ is countable (regarding $\mathbb{Q}^k$ as an infinite sequence with only the first k terms can be nonzero), which is countable.
- Holder's inequality is true only for $0 < p, q < 1$.
- You need to verify $\sup(f+g) \le \sup f + \sup g$.

2 Questions for last tutorial

1. True or false

   (a) If f is integrable on $[0,1]$, then f^2 is integrable on $[0,1]$.
   Ans: True.

   (b) If f^2 is integrable on $[0,1]$, then f is integrable on $[0,1]$.
   Ans: False, consider f to be the function $f(x) = 1$ when x rational and $f(x) = -1$ when x irrational.

   (c) If f^2 is integrable on $[0,1]$, then $|f|$ is integrable on $[0,1]$.
   Ans: True.

   (d) If f is non-negative and continuous on $(0,1]$, and $\int_0^1 f$ exists as an improper integral, then $\int_0^1 f^2$ exists as an improper integral.
   Ans: False, consider the function $f(x) = x^{-1/2}$.

   (e) If f is non-negative and continuous on $(0,1]$, and $\int_0^1 f^2$ exists as an improper integral, then $\int_0^1 f$ exists as an improper integral.
   Ans: True.

2. Let f be a function on $(-\pi,\pi]$, which is integrable on $[a,\pi]$ for any $a \in (-\pi,\pi]$, and that $\lim_{c\to-\pi}\int_c^\pi f$ exists, show that Riemann Lebesgue lemma holds.

Ans: For any ϵ, we can find $c > -\pi$ so that $\int_{-\pi}^{c} |f| < \epsilon$. Consider the function f' which is equal to f on $(c, \pi]$, and 0 on $[-\pi, c]$. By the usual Riemann Lebesgue lemma, $c_n(f') \to 0$, but we have $|c_n(f) - c_n(f')| < \epsilon/2\pi$.

3. If f is uniformly Lipschitz and 2π periodic, show that $c_n(f) = O(1/n)$. Discussed in tutorial.

4. Show that

$$-\log|2\sin\frac{x}{2}| \sim \sum_{n=1}^{\infty} \frac{\cos x}{n}$$

Hints: $\int_0^{\pi} \log\sin\frac{x}{2} = -\frac{\pi}{2}\log 2$.
Ans: It is an even function, so $b_n = 0\ \forall n$. a_0 can be calculated using the hint. For $n \geq 1$,

$$\begin{aligned}
\pi a_n &= -\int_{-\pi}^{\pi} \log(2\sin(x/2))\cos nx dx \\
&= -\frac{1}{n}\sin(nx)\log(2\sin(x/2))\Big|_{-\pi}^{\pi} + \frac{1}{n}\int_{-\pi}^{\pi} \frac{\sin(nx)\cos(x/2)}{2\sin(x/2)}dx \\
&= \frac{1}{n}\int_{-\pi}^{\pi} \frac{\sin(n+\frac{1}{2})x + \sin(n-\frac{1}{2})x}{4\sin(x/2)}dx \\
&= \frac{\pi}{2n}\int_{-\pi}^{\pi} D_n(x) + D_{n-1}(x)dx \\
&= \frac{\pi}{n}
\end{aligned}$$

3 Questions of this tutorial

1. True or False:

   (a) If $d : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be the function defined by $d(x, y) = 0$ if $x = y$ and $d(x, y) = |x| + |y|$, then d is a metric.

   (b) $d(f, g) = \int_0^1 |f - g|^2$ is a metric on $C([0, 1])$.

   (c) $d(f, g) = (\int_0^1 |f - g|^{1/2})^2$ is a metric on $C([0, 1])$.

   (d) (Bolzano-Weierstrass?) Consider $X = C([-\pi, \pi])$ with the L^2 metric. Any bounded sequence of functions in X (i.e. the norms of the functions are bounded by a common constant) has a Cauchy subsequence.

   (e) $(1-1 = 0?)$ Let E be a subset of a metric space X, then $X\backslash\overline{(X\backslash E)} = E^o$. $(E^{'-'} = E^o)$.

   (f) Let (X, d) be a metric space. Every closed subset of X is an intersection of open subsets of X.

   (g) Let (X, d) be a metric space. Every open subset of X is a union of closed subsets of X.

(h) Let (X, d) be a metric space, and $p \in X$. Then the closure of $\{x' \in X : d(x', x) < 1\}$ in X is $\{x' \in X : d(x', x) \leq 1\}$.

(i) Let (X, d) be a metric space. We say a subset E of X is dense if $\overline{E} = X$. If two continuous functions $f, g : X \to \mathbb{R}$ agree on a dense subset of X, then $f = g$.

(j) There is a metric on $\mathbb{R}$, so that every subset of $\mathbb{R}\backslash\{0\}$ is open, but $\{0\}$ is not open.

(k) Let (X, d) be a metric space, and suppose $X = \cup U_i$ with each U_i open. Then a function $f : X \to \mathbb{R}$ is continuous if and only if $f|_{U_i}$ is continuous for each i.

(l) Let (X, d) be a metric space, and suppose $X = \cup F_i$ with each F_i closed. Then a function $f : X \to \mathbb{R}$ is continuous if and only if $f|_{F_i}$ is continuous for each i.

2. Let p be a prime number, consider the following function $N_p : \mathbb{Q} \to \mathbb{R}$. Each nonzero rational number x can be written in the form

$$x = p^n \frac{a}{b}$$

with n an integer, and a, b are integers not divisible by p. We define $N_p(x) = p^{-n}$, and also define $N_p(0) = 0$.
Show that $d(x, y) = N_p(x-y)$ is a metric on $\mathbb{Q}$. Is the sequence $1, p, p^2, p^3, \ldots$ convergent?

3. Let A be an $n \times n$ matrix. We say A is symmetric if $A^T = A$. We say A is disconnected if we can find a subset I of $\{1, 2, \ldots, n\}$ such that $A_{ij} = 0$ whenever $i \in I, j \in J$. We also say that A is disconnected if A is connected.

(a) If A is connected, show that for any $i, j \in \{1, 2, \ldots, n\}$ there is some non negative integer k so that the (i, j) entry of A^k is nonzero. (By convention, $A^0 = I$).

(b) Assume A is symmetric and connected. For $i, j \in \{1, 2, \ldots, n\}$, define $d(i, j)$ to be the minimal non negative integer k so that the (i, j) entry of A^k is nonzero. Show that d is a metric.